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2009 J. Phys. A: Math. Theor. 42 105301

(http://iopscience.iop.org/1751-8121/42/10/105301)

J. Phys. A: Math. Theor. **42** (2009) 105301 (14pp) doi:10.1088/1751-8113/42/10/105301

A conjugate for the Bargmann representation

A D Ribeiro[1], **F Parisio**[2,3] **and M A M de Aguiar**[4]

[1] Departamento de Física, Universidade Federal do Paraná, 81531-990, Curitiba, PR, Brazil
[2] Departamento de Física, Universidade Federal de Pernambuco, 50670-901, Recife, PE, Brazil
[3] Instituto de Física, Universidade Federal de Alagoas, 57072-970, Maceió, Alagoas, Brazil.
[4] Instituto de Física 'Gleb Wataghin', Universidade Estadual de Campinas, Unicamp 13083-970, Campinas, São Paulo, Brazil

Received 9 September 2008
Published 13 February 2009
Online at stacks.iop.org/JPhysA/42/105301

Abstract

In the Bargmann representation of quantum mechanics, physical states are mapped into entire functions of a complex variable z^*, whereas the creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ play the role of multiplication and differentiation with respect to z^*, respectively. In this paper we propose an alternative representation of quantum states, conjugate to the Bargmann representation, where the roles of $\hat{a}^\dagger$ and $\hat{a}$ are reversed, much like the roles of the position and momentum operators in their respective representations. We derive expressions for the inner product that maintain the usual notion of distance between states in the Hilbert space. Applications to simple systems and to the calculation of semiclassical propagators are presented.

PACS numbers: 03.65.Db, 03.65.Ca, 03.65.Sq

1. Introduction

In non-relativistic quantum mechanics, the position and the momentum of a particle are represented by operators $\hat{q}$ and $\hat{p}$ satisfying the canonical commutation relation $[\hat{q}, \hat{p}] = \mathrm{i}\hbar$. Although both operators have a continuous spectrum and are unbounded, the generalized eigenstates of $\hat{q}$, obeying $\hat{q}|q\rangle = q|q\rangle$ and the Dirac delta normalization $\langle q'|q\rangle = \delta(q-q')$ [1], provide a resolution of the unit operator and define the *coordinate representation*, where state kets $|\psi\rangle$ in the Hilbert space are mapped into square-integrable wavefunctions $\psi(q) = \langle q|\psi\rangle$ in $L^2(\mathbf{R})$ with $\hat{q}|\psi\rangle \to \langle q|\hat{q}|\psi\rangle = q\psi(q)$ and $\hat{p}|\psi\rangle \to \langle q|\hat{p}|\psi\rangle = -\mathrm{i}\hbar\partial\psi(q)/\partial q$[5]. Similarly, the generalized eigenstates of $\hat{p}$ define the *momentum representation*, where $|\psi\rangle \to \tilde{\psi}(p) = \langle p|\psi\rangle$ with $\hat{p}|\psi\rangle \to \langle p|\hat{p}|\psi\rangle = p\tilde{\psi}(p)$ and $\hat{q}|\psi\rangle \to \langle p|\hat{q}|\psi\rangle = \mathrm{i}\hbar\partial\tilde{\psi}(p)/\partial p$. The

[5] We refer to [1], specially chapters 3 and 6, for a discussion of the many subtleties related to unbounded operators and their associated representations, including classes of equivalence of functions in $L^2(\mathbf{R})$.

two representations are said to be conjugate to each other and are related by the Fourier transformation

$$\psi(q) = \int \langle q|p\rangle \tilde{\psi}(p)\,\mathrm{d}p = \frac{1}{\sqrt{2\pi\hbar}} \int \tilde{\psi}(p)\,\mathrm{e}^{\mathrm{i}pq/\hbar}\,\mathrm{d}p. \tag{1}$$

The interplay between the position and the momentum representations is of great importance in the quantum theory. Although the information contained in either representation is the same, the clarity and simplicity of a calculation depend strongly on which representation is chosen. Simple illustrations in one dimension are the time-independent Schrödinger equation for the square barrier potential, which is very simple in the coordinate representation, and the linear potential $V(q) = q$, which can be solved immediately in the momentum representation. This is particularly useful to calculate WKB wavefunctions near turning points, where the coordinate representation is singular.

More elaborate applications involve semiclassical approximations for dynamical processes, such as the propagator $K(q_f, q_i, T)$, which depends on classical trajectories starting at q_i and ending at q_f after a time T. The semiclassical approximation for K diverges at the so-called focal points, where $\partial p_i/\partial q_f \to \infty$, and is inaccurate in a whole vicinity of these points [2]. In general, a focal point in the position representation is not a simultaneous focal point in the momentum representation and, as proposed by Maslov [3, 4], one can switch between the two representations to pass by the focal point. In other words, the semiclassical approximation for $K(p_f, q_i, T)$ is well behaved when calculated at the same trajectory where $K(q_f, q_i, T)$ is divergent, and can be Fourier transformed to produce accurate results for the original propagator $K(q_f, q_i, T)$.

Besides the position and momentum representations, a different set of continuous basis states can be defined with the help of coherent states, whose importance in physics has been recognized since the early days of quantum mechanics [5–10]. In the special case of the harmonic oscillator, coherent states are associated with the creation and annihilation operators

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{\hat{q}}{b} + \mathrm{i}\frac{\hat{p}}{c}\right), \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{\hat{q}}{b} - \mathrm{i}\frac{\hat{p}}{c}\right), \tag{2}$$

where $b = \sqrt{\hbar/(m\omega)}$ and $c = \sqrt{m\hbar\omega}$, with m and ω the mass and frequency of the oscillator, respectively. The commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, together with the eigenvalue equation $\hat{a}|z\rangle = z|z\rangle$, define an alternative representation of quantum mechanics which was introduced by Fock and studied in detail by Bargmann [5], who lent his name to the theory, Glauber [6] and others [7–10]. In the *Bargmann* representation, the state $|\psi\rangle$ is mapped into an entire function $\psi(z^*) = \langle z|\psi\rangle$ of the complex variable

$$z^* = \frac{1}{\sqrt{2}}\left(\frac{q}{b} - \mathrm{i}\frac{p}{c}\right), \tag{3}$$

where $\hat{a}^\dagger|\psi\rangle \to \langle z|\hat{a}^\dagger|\psi\rangle = z^*\psi(z^*)$ and $\hat{a}|\psi\rangle \to \langle z|\hat{a}|\psi\rangle = \partial\psi(z^*)/\partial z^*$. The real numbers $q = \langle z|\hat{q}|z\rangle/\langle z|z\rangle$ and $p = \langle z|\hat{p}|z\rangle/\langle z|z\rangle$ in equation (3) are the average values of the position and momentum operators respectively.

The (unnormalized) Bargmann states $|z\rangle$ are given by

$$|z\rangle = \mathrm{e}^{z\hat{a}^\dagger}|0\rangle, \tag{4}$$

where $|0\rangle$ is the ground state of the harmonic oscillator. These states are related to the (normalized) canonical coherent states $|z\rangle\rangle$ by $|z\rangle\rangle = \mathrm{e}^{-|z|^2/2}|z\rangle$.

Two comments are in order here: first, we note that both $|z\rangle$ and $|z\rangle\rangle$ are square integrable, although only the latter is normalized. Indeed, $\langle z|z\rangle = \mathrm{e}^{|z|^2} < \infty$. The reason for moving the normalization factor $\mathrm{e}^{-|z|^2/2}$ from $|z\rangle\rangle$ to the integration measure (see the following section) is to ensure that $\langle z|\psi\rangle$ is an analytic function of z^*. Second, the definition of coherent states as eigenstates of the annihilation operator, as proposed by Glauber [6], does not generalize to finite Hilbert spaces (or to compact Lie groups) [10]. In these cases coherent states have to be defined by group theoretical methods [9, 10]. However, as pointed out by Zhang *et al* [10], for the Weyl group or field coherent states, there is no obvious advantages in using the group-theoretical approach and we adopt the more intuitive Glauber's definition.

In contrast to the position and the momentum representations, the Bargmann representation lacks a dual counterpart. Indeed, since the operator $\hat{a}^\dagger$ does not have eigenstates, it is not clear whether $|\psi\rangle$ can be mapped into $\tilde{\psi}(z)$ such that $\hat{a}|\psi\rangle$ is mapped into $z\tilde{\psi}(z)$. It might be argued that, because it is a phase-space representation, where both q and p participate simultaneously, a conjugate representation is simply not needed. It was initially thought, for instance, that the phase-space propagator $K(z_f^*, z_i, T) = \langle z_f|\mathrm{e}^{-\mathrm{i}\hat{H}T/\hbar}|z_i\rangle$ would be free of focal points [11–14]. Focal points, however, do exist in the coherent state propagator [15–18] and in mixed representations as well [19–23] and the application of the Maslov method would require a conjugate representation for the Bargmann states.

The existence of phase-space focal points motivated the definition of an application that could play the role of a conjugate representation for the Bargmann states [24] and that was successfully used in applications of the Maslov method [25, 26]. Its relation to the Bargmann representation, however, is not as simple as the relation between the position and momentum representations, but it does comply with the basic requirements of a dual map. The purpose of this paper is to formalize this conjugate representation and to study it in more detail.

The paper is organized as follows: in section 2 we review some of the main ingredients of the Bargmann representation. In section 3 we define its conjugate counterpart in terms of line integrals in the complex plane and study some of its properties. In section 4 we present alternative formulae where the line integrals are replaced by integrals over the entire complex plane and in section 5 we show a few simple applications. Finally, in section 6, we summarize our results.

2. The Bargmann representation

In the Bargmann formalism, a state ket $|\psi\rangle$ is represented in phase space by its projection onto a non-normalized coherent state

$$\psi(z^*) = \langle z|\psi\rangle = \langle 0|\,\mathrm{e}^{z^*a}|\psi\rangle. \tag{5}$$

The state of the system is completely determined by the entire function $\psi(z^*)$. The resolution of unit is expressed in terms of the integral

$$\hat{I} = \int \frac{\mathrm{d}^2 z}{\pi}\,\mathrm{e}^{-|z|^2}|z\rangle\langle z| \equiv \int \frac{\mathrm{d}q\,\mathrm{d}p}{2\pi\hbar}\,\mathrm{e}^{-|z|^2}|z\rangle\langle z| \equiv \int \mathrm{d}^2\mu(z)|z\rangle\langle z| \tag{6}$$

so that the inner product between $|\psi\rangle$ and $|\phi\rangle$ reads

$$\langle\psi|\phi\rangle = \int \mathrm{d}^2\mu(z)\psi^*(z^*)\phi(z^*) \equiv (\psi,\phi), \tag{7}$$

where the last equality also defines the inner product between the two corresponding entire functions.

The Bargmann space $\mathcal{F}$ is composed of entire functions $\psi(z^*)$ such that $(\psi,\psi) < \infty$. The mapping between $\psi(q)$ and $\psi(z^*)$ can be constructed explicitly as

$$\psi(z^*) = \int \mathrm{d}q\,\langle z|q\rangle\langle q|\psi\rangle = \pi^{-1/4}b^{-1/2}\int \mathrm{d}q\,\mathrm{e}^{-\frac{1}{2}(z^{*2}+q^2/b^2)+\sqrt{2}z^*q/b}\psi(q) \tag{8}$$

with its inverse given by

$$\psi(q) = \pi^{-1/4} b^{-1/2} \int \mathrm{d}\mu(z)\, \mathrm{e}^{-\frac{1}{2}(z^2+q^2/b^2)+\sqrt{2}zq/b} \psi(z^*). \tag{9}$$

If ψ and ϕ are expressed as a power series as

$$\psi(z^*) = \sum_{n=0}^{\infty} a_n z^{*n}/\sqrt{n!}, \qquad \phi(z^*) = \sum_{n=0}^{\infty} b_n z^{*n}/\sqrt{n!} \tag{10}$$

the overlap reduces to

$$\langle\psi|\phi\rangle = \sum_{n=0}^{\infty} a_n^* b_n. \tag{11}$$

Therefore, the set of functions

$$\phi_n(z^*) = \langle z|n\rangle = z^{*n}/\sqrt{n!} \tag{12}$$

forms a complete orthonormal set in $\mathcal{F}$, where $|n\rangle$ are the eigenstates of the underlying harmonic oscillator, equations (2)–(4).

In the Bargmann representation it is convenient to express observables in terms of creation and annihilation operators. The action of these operators on $|\psi\rangle$ yields

$$\langle z|\hat{a}^\dagger|\psi\rangle = z^*\psi(z^*), \qquad \langle z|\hat{a}|\psi\rangle = \frac{\partial}{\partial z^*}\psi(z^*). \tag{13}$$

Any observable $\hat{A}(\hat{a}^\dagger, \hat{a})$ is, therefore, written in the Bargmann representation as $\hat{A}_B = \hat{A}\big(z^*, \frac{\partial}{\partial z^*}\big)$. This identification is valid for any ordering of the operators since the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ is preserved, i.e., $\big[\frac{\partial}{\partial z^*}, z^*\big] = 1$. Thus, one can recast the time-independent Schrödinger equation $\hat{H}|\psi\rangle = E|\psi\rangle$ as $\hat{H}_B\psi(z^*) = E\psi(z^*)$. For the simple harmonic oscillator $\hat{H}_B = \hbar\omega\big(z^*\frac{\partial}{\partial z^*} + 1/2\big)$ and we get

$$\hbar\omega\left(z^*\frac{\partial}{\partial z^*} + \frac{1}{2}\right) u_n(z^*) = E_n u_n(z^*), \tag{14}$$

whose solutions are exactly the normalized functions $\phi_n(z^*)$, defined by equation (12), with eigenvalues $E_n = \hbar\omega(n + 1/2)$, $n = 0, 1, 2, \cdots$.

Before closing this section we derive Bargmann's reproducing kernel from the resolution of unit (6). Multiplying this equation on the right by $|\psi\rangle$ and on left by $\langle w|$ we obtain

$$\psi(w^*) = \int \mathrm{d}^2\mu(z)\langle w|z\rangle\psi(z^*) = \int \mathrm{d}^2\mu(z)\, \mathrm{e}^{w^*z}\psi(z^*). \tag{15}$$

The reproducing kernel $\mathcal{K}(w^*, z) = \langle w|z\rangle = \mathrm{e}^{w^*z}$ plays the role of the delta function in the position and momentum representations and will be important to derive some useful relations in the following sections.

3. The conjugate application

3.1. Basic definitions

Let $|\psi\rangle$ be a state ket and $\psi(z^*) = \langle z|\psi\rangle$ its Bargmann representation. For each coherent state $|w\rangle$ we define the application $|\psi\rangle \longrightarrow f_\psi(w)$ by

$$f_\psi(w) = \int_\gamma \frac{\langle z|\psi\rangle}{\langle z|w\rangle}\, \mathrm{d}z^* = \int_\gamma \psi(z^*)\, \mathrm{e}^{-z^*w}\, \mathrm{d}z^* \tag{16}$$

and its inverse by

$$\psi(z^*) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma'} f_\psi(w) \langle z|w\rangle \, \mathrm{d}w = \frac{1}{2\pi i} \int_{\gamma'} f_\psi(w) \, \mathrm{e}^{z^* w} \, \mathrm{d}w. \tag{17}$$

The integration paths γ and γ' will be defined below.

Although the denominator in equation (16) might look unusual, it is really a direct generalization of the transformation between the coordinate and the momentum representations, which can be written as

$$\tilde{\psi}(p) = \frac{1}{2\pi\hbar} \int \frac{\langle q|\psi\rangle}{\langle q|p\rangle} \, \mathrm{d}q \qquad \text{and} \qquad \psi(q) = \int \langle q|p\rangle\langle p|\psi\rangle \, \mathrm{d}p.$$

However, while both $\psi(q)$ and $\tilde{\psi}(p)$ are matrix elements between the ket $|\psi\rangle$ and a bra, $f_\psi(w)$ is not itself a matrix element. Moreover, the application is linear in $|\psi\rangle$, since

$$f_{\alpha\psi+\beta\phi}(w) = \alpha f_\psi + \beta f_\phi, \tag{18}$$

but not in $|w\rangle$. For this reason the nomenclature *conjugate application* is preferred instead of conjugate representation.

3.2. Action of operators

Before we specify the integration paths γ and γ' we explore the action of operators on the conjugate functions. Consider two states $|\psi_1\rangle = \hat{a}^\dagger|\psi\rangle$ and $|\psi_2\rangle = \hat{a}|\psi\rangle$, whose Bargmann representations are given, respectively, by $\psi_1(z^*) = z^*\psi(z^*)$ and $\psi_2(z^*) = \frac{\partial\psi(z^*)}{\partial z^*}$. The corresponding conjugate functions are, according to (16),

$$f_{\psi_1}(w) = \int_\gamma z^*\psi(z^*) \, \mathrm{e}^{-z^* w} \, \mathrm{d}z^* = -\frac{\partial}{\partial w} f_\psi(w) \tag{19}$$

and

$$f_{\psi_2}(w) = \int_\gamma \frac{\partial\psi(z^*)}{\partial z^*} \, \mathrm{e}^{-z^* w} \, \mathrm{d}z^* = w f_\psi(w), \tag{20}$$

where we have integrated by parts and assumed that $\psi(z^*)\,\mathrm{e}^{-z^* w}$ vanishes at the extremes of γ (see comment after equation (32) in the following subsection). Consequently, if $|\phi\rangle = \hat{A}(\hat{a}, \hat{a}^\dagger)|\psi\rangle$ and $\phi(z^*) = \hat{A}_B\big(\frac{\partial}{\partial z^*}, z^*\big)\psi(z^*)$, then

$$f_\phi(w) = \hat{A}\left(w, -\frac{\partial}{\partial w}\right) f_\psi(w) \equiv \hat{A}_C f_\psi(w) \tag{21}$$

since the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ is preserved in the form $[w, -\partial/\partial w] = 1$.

The duality between the two representations is therefore expressed by the action of $\hat{a}$ and $\hat{a}^\dagger$ on the corresponding functions $\psi(z^*)$ and $f_\psi(w)$:

$$\begin{array}{ccccc} \hat{a} & \xrightarrow{\text{Bargmann}} & \dfrac{\partial}{\partial z^*} & \xrightarrow{\text{conjugate}} & w \\ \hat{a}^\dagger & \xrightarrow{\text{Bargmann}} & z^* & \xrightarrow{\text{conjugate}} & -\dfrac{\partial}{\partial w}. \end{array} \tag{22}$$

In particular, the Schrödinger equation in the space of functions $f_\psi(w)$ becomes

$$\hat{H}_C\left(-\frac{\partial}{\partial w}, w\right) f_\psi(w) = E f_\psi(w). \tag{23}$$

For the harmonic oscillator we obtain

$$\hbar\omega\left(-\frac{\partial}{\partial w} w + \frac{1}{2}\right) f_\psi(w) = E f_\psi(w) \tag{24}$$

and the eigenfunctions and eigenvalues can immediately be calculated as

$$f_n(w) = \frac{\sqrt{n!}}{w^{n+1}}, \qquad E_n = \hbar\omega(n + 1/2), \tag{25}$$

where the choice of normalization is justified in the following subsection.

3.3. Integration paths

In order to define the paths γ and γ' in equations (16) and (17), we consider the expansion of a general ket $|\psi\rangle$ in the harmonic oscillator basis $\{|n\rangle\}$, namely, $|\psi\rangle = \sum_{n=0}^{\infty} a_n|n\rangle$. The Bargmann representation of $|\psi\rangle$ is

$$\psi(z^*) = \sum_{n=0}^{\infty} a_n\langle z|n\rangle = \sum_{n=0}^{\infty} a_n\phi_n(z^*) = \sum_{n=0}^{\infty} \frac{a_n z^{*n}}{\sqrt{n!}}, \tag{26}$$

where

$$a_n = \frac{1}{\sqrt{n!}} \int \psi(z^*)z^n \,\mathrm{d}^2\mu(z). \tag{27}$$

Inserting (26) into equation (16), we find

$$f_\psi(w) = \sum_{n=0}^{\infty} a_n f_{\phi_n}(w), \tag{28}$$

where

$$f_{\phi_n}(w) = \int_\gamma \phi_n(z^*)\,\mathrm{e}^{-z^*w}\,\mathrm{d}z^* = \frac{1}{\sqrt{n!}} \int_\gamma z^{*n}\,\mathrm{e}^{-z^*w}\,\mathrm{d}z^*. \tag{29}$$

We now demand that $f_{\phi_n}(w) = f_n(w)$, given by equation (25). This is achieved by converting the line integral into a Laplace transform. Writing z and w in terms of polar variables, $z = r_z\,\mathrm{e}^{\mathrm{i}\theta_z}$ and $w = r_w\,\mathrm{e}^{\mathrm{i}\theta_w}$, the exponent of the integrand becomes $-z^*w = -r_z r_w\,\mathrm{e}^{\mathrm{i}(\theta_w-\theta_z)}$. The path γ is fixed by choosing $\theta_z = \theta_w$ and r_z going from 0 to ∞. In fact, since the function being integrated is analytic, it suffices to take paths that can be deformed into this one. Explicitly we obtain

$$f_{\phi_n}(w) = \frac{\mathrm{e}^{-\mathrm{i}(n+1)\theta_w}}{\sqrt{n!}} \int_0^\infty r_z^n\,\mathrm{e}^{-r_z r_w}\,\mathrm{d}r_z = \frac{w^{-(n+1)}}{\sqrt{n!}}\Gamma(n+1) = \frac{\sqrt{n!}}{w^{n+1}}, \tag{30}$$

which leads to the Laurent series

$$f_\psi(w) = \sum_{n=0}^{\infty} a_n \frac{\sqrt{n!}}{w^{n+1}}. \tag{31}$$

Provided the sum on the right side converges we can also write

$$a_n = \frac{1}{\sqrt{n!}} \int f_\psi(w)w^{n+1}\,\mathrm{d}^2\mu(w). \tag{32}$$

This choice of γ also guarantees the correctness of equation (20) for functions that can be expressed as power series like (26), since $\partial\psi/\partial z^*$ does not depend on a_0.

Alternatively, using this integration path directly into equation (16) leads to

$$f_\psi(w) = \int_0^\infty \psi(r_z\,\mathrm{e}^{-\mathrm{i}\theta_w})\,\mathrm{e}^{-r_z r_w - \mathrm{i}\theta_w}\,\mathrm{d}r_z = \frac{1}{w}\int_0^\infty \psi\left(\frac{x}{w}\right)\mathrm{e}^{-x}\,\mathrm{d}x. \tag{33}$$

Similarly, the inverse transform of $f_{\phi_n}(w)$ can be written as

$$\frac{1}{2\pi\mathrm{i}} \int_{\gamma'} f_{\phi_n}(w)\,\mathrm{e}^{wz^*}\,\mathrm{d}w = \frac{\sqrt{n!}}{2\pi\mathrm{i}} \int_{\gamma'} \frac{\mathrm{e}^{wz^*}}{w^{n+1}}\,\mathrm{d}w, \tag{34}$$

which can be performed by using Cauchy's residue theorem, when γ' is conveniently chosen and the integral becomes a Mellin integral. Since the pole is located at the origin, γ' should be perpendicular to the straight line connecting the origin with z, crossing the real axis on the positive (negative) side if $\mathrm{Re}(z^*) > 0$ ($\mathrm{Re}(z^*) < 0$). Then, we get

$$\frac{1}{2\pi\mathrm{i}}\int_{\gamma'} f_{\phi_n}(w)\,\mathrm{e}^{wz^*}\,\mathrm{d}w = \frac{1}{\sqrt{n!}}\left(\frac{\mathrm{d}^n \mathrm{e}^{wz^*}}{\mathrm{d}w^n}\right)\bigg|_{w=0} = \frac{z^{*n}}{\sqrt{n!}} = \phi_n(z^*). \tag{35}$$

Therefore, the general inverse formula can be written as

$$\psi(z^*) = \frac{1}{2\pi z^*}\int_{-\infty-i\epsilon}^{\infty-i\epsilon} f_\psi\left(\frac{\mathrm{i}v}{z^*}\right)\mathrm{e}^{\mathrm{i}v}\,\mathrm{d}v, \tag{36}$$

where ϵ is a positive number.

Alternative expressions for the mappings between $\psi(z^*)$ and $f_\psi(w)$ that avoid the line integrals will be given in section 4.

3.4. Scalar product

The scalar product between two kets $|\psi\rangle$ and $|\phi\rangle$ can be obtained starting from

$$\langle\psi|\phi\rangle = \int \psi^*(z^*)\phi(z^*)\,\mathrm{d}^2\mu(z) = \sum_{n,m} a_m^* b_n \int \phi_m^*(z^*)\phi_m(z^*)\,\mathrm{d}^2\mu(z), \tag{37}$$

where $|\psi\rangle = \sum_{n=0}^{\infty} a_n|n\rangle$ and $|\phi\rangle = \sum_{n=0}^{\infty} b_n|n\rangle$. Using equations (35) and (36) we obtain

$$\langle\psi|\phi\rangle = \frac{1}{4\pi^2}\sum_{n,m=0}^{\infty} a_m^* b_n \sqrt{m!n!}\mathcal{A}_{mn}, \tag{38}$$

where

$$\mathcal{A}_{mn} = \int\left[z^{*m}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\epsilon+\mathrm{i}y}}{(\epsilon+\mathrm{i}y)^{m+1}}\,\mathrm{d}y\right]^*\left[z^{*n}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\epsilon+\mathrm{i}y}}{(\epsilon+\mathrm{i}y)^{n+1}}\mathrm{d}y\right]\mathrm{d}^2\mu(z). \tag{39}$$

The integration over $\mathrm{d}^2\mu(z)$ gives

$$\int z^n z^{*m}\,\mathrm{d}^2\mu(z) = m!\delta_{mn}. \tag{40}$$

In addition, the integral inside the first brackets of equation (39) can be evaluated by residues resulting in $2\pi/m!$. Thus,

$$\begin{aligned}\mathcal{A}_{mn} &= 2\pi\delta_{mn}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\epsilon+\mathrm{i}y}}{(\epsilon+\mathrm{i}y)^{n+1}}\,\mathrm{d}y\\ &= 2\pi\delta_{mn}\int_0^{\infty}\left[\frac{\mathrm{e}^{\epsilon+\mathrm{i}y}}{(\epsilon+\mathrm{i}y)^{n+1}}+\frac{\mathrm{e}^{\epsilon-\mathrm{i}y}}{(\epsilon-\mathrm{i}y)^{n+1}}\right]\mathrm{d}y\end{aligned} \tag{41}$$

and equation (38) becomes

$$\langle\psi|\phi\rangle = \frac{1}{2\pi}\sum_{n,m=0}^{\infty} a_m^* b_n\sqrt{m!n!}\delta_{mn}\int_0^{\infty}\left[\frac{e^{\epsilon+\mathrm{i}y}}{(\epsilon+\mathrm{i}y)^{n+1}}+\frac{e^{\epsilon-\mathrm{i}y}}{(\epsilon-\mathrm{i}y)^{n+1}}\right]\mathrm{d}y. \tag{42}$$

At last, we use the identity

$$\frac{\delta_{mn}\,\mathrm{e}^{\epsilon\pm\mathrm{i}y}}{(\epsilon\pm\mathrm{i}y)^{n+1}} = \frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{e}^{\mathrm{i}(m-n)\theta}\,\mathrm{e}^{\epsilon\pm\mathrm{i}y}\,\mathrm{d}\theta}{(\epsilon\pm\mathrm{i}y)^{\frac{n+1}{2}}(\epsilon\pm\mathrm{i}y)^{\frac{m+1}{2}}} \tag{43}$$

and change the variable of integration in equation (42) as $y = r^2$ obtaining

$$\langle\psi|\phi\rangle = \frac{1}{2\pi^2}\int_0^\infty r\,\mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta\,\mathrm{F}_\epsilon(r,\theta), \tag{44}$$

where

$$\mathrm{F}_\epsilon(r,\theta) = f_\psi^*(w_{(+)}\,\mathrm{e}^{-\mathrm{i}\pi/4})\,f_\phi(w_{(-)}\,\mathrm{e}^{+\mathrm{i}\pi/4})\,\mathrm{e}^{+\mathrm{i}(r^2-\mathrm{i}\epsilon)} + f_\psi^*(w_{(-)}\,\mathrm{e}^{+\mathrm{i}\pi/4})\,f_\phi(w_{(+)}\,\mathrm{e}^{-\mathrm{i}\pi/4})\,\mathrm{e}^{-\mathrm{i}(r^2+\mathrm{i}\epsilon)} \tag{45}$$

and $w_{(\pm)} = \sqrt{r^2 \pm \mathrm{i}\epsilon}\,\mathrm{e}^{\mathrm{i}\theta}$. Finally, taking the limit $\epsilon \to 0$, $w_{(+)} = w_{(-)} = r\mathrm{e}^{\mathrm{i}\theta}$ and, defining $w_1 = w\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}$ and $w_2 = w\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$ we obtain

$$\langle\psi|\phi\rangle = \frac{1}{\pi^2}\int\left[f_\psi^*(w_1)\,f_\phi(w_2)\,\mathrm{e}^{\mathrm{i}|w|^2} + f_\psi^*(w_2)\,f_\phi(w_1)\,\mathrm{e}^{-\mathrm{i}|w|^2}\right]\mathrm{d}^2w. \tag{46}$$

4. Alternative formal transformations

4.1. The coherent state

The Weyl displacement operator $\hat{D} = \mathrm{e}^{z\hat{a}^\dagger}$ in the dual space becomes $\hat{D}_C = \mathrm{e}^{-z\frac{\partial}{\partial w}}$. As a consequence, since $|z\rangle = \mathrm{e}^{z\hat{a}^\dagger}|0\rangle = \hat{D}|0\rangle$ and the conjugate of the ground state is $f_0(w) = \frac{1}{w}$, we find

$$f_z(w) = \mathrm{e}^{-z\frac{\partial}{\partial w}}\left(\frac{1}{w}\right) = \frac{1}{w} + \frac{z}{w^2} + \frac{z^2}{w^3} + \cdots = \frac{1}{w-z} \tag{47}$$

for $|z/w| < 1$. This shows that $\hat{D}_C$ also acts as a displacement operator in the dual space. Although the series does not converge inside the circle $|z/w| = 1$, we analytically extend it to $f_z(w) = 1/(w-z)$ to the whole complex plane, except for $w = z$. This continuation is justified because the path of integration γ' in the inverse transformation (17) can always be chosen to lie outside the circle $|z/w| = 1$ and, therefore, the integral is independent of $f_z(w)$ in this region. Note that the result (47) can also be obtained using the basic definition (33). This time the convergence region is for w outside the circle of radius $|z|/2$ centered on $z/2$, which is less restrictive than that obtained via displacement operator. Analytic continuation is then similar to that done for the Laplace transform of the exponential function.

Equation (47) provides an important formal expression of $f_\psi(w)$. Starting from the expansion for $|\psi\rangle$ in coherent states

$$|\psi\rangle = \int\langle z|\psi\rangle|z\rangle\,\mathrm{d}^2\mu(z) \tag{48}$$

and transforming both sides we obtain

$$f_\psi(w) = \int\psi(z^*)\,f_z(w)\,\mathrm{d}^2\mu(z). \tag{49}$$

The integral in equation (49) is over the whole phase space, avoiding the cumbersome line integrals of the original definition. However, this expression is only formal, since going from (48) to (49) involves the illegal interchange of the line integral coming from the definition of f_ψ and the integral over the complex plane from (48). Nevertheless, expanding $|z\rangle$ in (48) in the harmonic oscillator basis states and doing the integral term by term we obtain the convergent expression

$$f_\psi(w) = \sum_{n=0}^\infty\frac{1}{w^{n+1}}\int z^n\,\psi(z^*)\,\mathrm{d}^2\mu(z). \tag{50}$$

This procedure is equivalent to treating $f_z(w)$ formally as the series given by (47) and interchange the summation and integration.

Although the direct transformation (49) is only formal and essentially useless, it is possible to write the inverse transformation in the same footing which is valid for all $f_\psi(w)$. Using the formal expression (49) temporarily we write

$$\psi(z^*) = \int A(z^*, t) f_\psi(t)\, \mathrm{d}^2\mu(t) = \int A(z^*, t)\psi(w^*) f_w(t)\, \mathrm{d}^2\mu(w)\, \mathrm{d}^2\mu(t). \tag{51}$$

Comparing with the reproducing kernel equation (15) we find that

$$\int A(z^*, t) f_w(t)\, \mathrm{d}^2\mu(t) = \mathrm{e}^{z^* w}. \tag{52}$$

Going back to the series representation for $f_w(t)$ and exchanging the summation and integration we recast this equation as

$$\sum_{n=0}^{\infty} w^n \left[\int \frac{A(z^*, t)}{t^{n+1}}\, \mathrm{d}^2\mu(t)\right] = \mathrm{e}^{z^* w}, \tag{53}$$

which shows that

$$\int \frac{A(z^*, t)}{t^{n+1}}\, \mathrm{d}^2\mu(t) \equiv \frac{z^{*n}}{n!}. \tag{54}$$

Differentiating both sides n times with respect to z^* gives one. Comparing again with equation (15) we find that $A(z^*, t) = t \exp\{z^* t\}$ and

$$\psi(z^*) = \int w\, \mathrm{e}^{z^* w} f_\psi(w)\, \mathrm{d}^2\mu(w). \tag{55}$$

In contrast to (49), this equation is well defined for all $f_\psi(w)$. This is an interesting expression that allows the construction of matrix elements, such as propagators, from the usual phase-space integration of their dual forms. We show in the appendix how to perform the integral for the basic cases $|\psi\rangle = |n\rangle$ and $|\psi\rangle = |z\rangle$.

4.2. Reproducing kernel

When equation (55) is substituted back into (49) we get

$$\begin{aligned} f_\psi(w) &= \int w'\, \mathrm{e}^{z^* w'} f_\psi(w') f_z(w)\, \mathrm{d}^2\mu(w')\, \mathrm{d}^2\mu(z). \\ &\equiv \int \mathcal{K}_C(w, w') f_\psi(w')\, \mathrm{d}^2\mu(w'). \end{aligned} \tag{56}$$

Comparing with (15) we find

$$\mathcal{K}_C(w, w') = \int w'\, \mathrm{e}^{z^* w'} f_z(w)\, \mathrm{d}^2\mu(z) = w' f_{w'}(w) = \frac{w'}{w - w'}. \tag{57}$$

Once again these expressions are only formal and the operational reproducing equation is

$$f_\psi(w) = \sum_{n=0}^{\infty} \frac{1}{w^{n+1}} \int w'^{n+1} f_\psi(w')\, \mathrm{d}^2\mu(w'). \tag{58}$$

4.3. Scalar product

An expression for the scalar product can be obtained from equations (7) and (55):

$$
\begin{aligned}
(\psi, \phi) &= \int \mathrm{d}^2\mu(z)\psi^*(z^*)\phi(z^*) \\
&= \int t^* w\, \mathrm{e}^{zt^*+z^*w} f_\psi^*(t) f_\phi(w)\, \mathrm{d}^2\mu(t)\, \mathrm{d}^2\mu(w)\, \mathrm{d}^2\mu(z).
\end{aligned} \tag{59}
$$

Using (15) again we find

$$
\int \mathrm{e}^{zt^*+z^*w}\, \mathrm{d}^2\mu(z) = \mathrm{e}^{t^*w} \tag{60}
$$

and

$$
(\psi, \phi) = \int t^* w\, \mathrm{e}^{t^*w} f_\psi^*(t) f_\phi(w)\, \mathrm{d}^2\mu(t)\, \mathrm{d}^2\mu(w). \tag{61}
$$

We can check the correctness of this expression by expanding the exponential in power series and rewriting this as

$$
\begin{aligned}
(\psi, \phi) &= \sum_{n=0}^{\infty} \left[\int \frac{t^{n+1} f_\psi(t)}{\sqrt{n!}} \mathrm{d}^2\mu(t)\right]^* \left[\int \frac{w^{n+1} f_\phi(w)}{\sqrt{n!}} \mathrm{d}^2\mu(w)\right] \\
&= \sum_{n=0}^{\infty} a_n^* b_n,
\end{aligned} \tag{62}
$$

where we used (32) with a_n and b_n as coefficients for ψ and ϕ respectively (10). A mixed representation for the scalar product can also be obtained by combining equations (61) and (55)

$$
(\psi, \phi) = \int t^* f_\psi^*(t)\phi(t^*)\, \mathrm{d}^2\mu(t). \tag{63}
$$

This expression might be useful, considering that for the eigenstates of the harmonic oscillator $t^* f_{\phi_n}^*(t)\phi_n(t^*) = 1$.

5. Simple examples

5.1. The propagator of the harmonic oscillator

In the Bargmann representation the propagator of the harmonic oscillator is [24]

$$
k(z^*, z_0, t) \equiv \langle z|\, \mathrm{e}^{-\mathrm{i}\hat{H}t/\hbar} |z_0\rangle = \mathrm{e}^{z_0(t)z^* - \mathrm{i}\omega t/2}, \tag{64}
$$

where $z_0(t) = z_0\, \mathrm{e}^{-\mathrm{i}\omega t}$. Its conjugate representation becomes

$$
f_k(w, z_0, t) = \mathrm{e}^{-\mathrm{i}\omega t/2} f_{z_0(t)}(w) = \frac{\mathrm{e}^{-\mathrm{i}\omega t/2}}{w - z_0(t)}. \tag{65}
$$

The diagonal conjugate representation becomes simply

$$
f_k(w, w, t) = \frac{1}{w} \sum_{n=0}^{\infty} \mathrm{e}^{-\mathrm{i}\omega(n+1/2)t}, \tag{66}
$$

which corresponds directly to its decomposition in eigenfunctions.

5.2. Position and momentum eigenstates

Although $\langle z|q\rangle$ does not belong to the Bargmann space $\mathcal{F}$ of square integrable functions, we can readily write it as

$$\langle z|q\rangle = \sum_{n=0}^{\infty}\langle z|n\rangle\langle n|q\rangle = \sum_{n=0}^{\infty}\frac{z^{*n}}{\sqrt{n!}}\phi_n(q), \tag{67}$$

where

$$\phi_n(q) = \frac{\pi^{-1/4}b^{-1/2}}{2^{n/2}\sqrt{n!}}\,\mathrm{e}^{-q^2/2b^2}H_n(q/b), \tag{68}$$

$b = \sqrt{\hbar/m\omega}$ and H_n are the Hermite polynomials. When (68) is placed into (67) the sum can be performed an results in the well-known expression

$$\langle z|q\rangle = \pi^{-1/4}b^{-1/2}\exp\left\{-\frac{q^2}{2b^2} - \frac{z^{*2}}{2} + \frac{\sqrt{2}z^*q}{b}\right\}. \tag{69}$$

The expression for $|q\rangle$ in the conjugate representation can be obtained directly from equations (68) and (67) and results in

$$f_q(w) = \pi^{-1/4}b^{-1/2}\,\mathrm{e}^{-q^2/2b^2}\sum_{n=0}^{\infty}\frac{1}{w^{n+1}}2^{-n/2}H_n(q/b). \tag{70}$$

Alternatively, using the integral form given by equation (36) we find, for $\mathrm{Re}(w^2) > 0$,

$$f_q(w) = \frac{\pi^{1/4}b^{-1/2}}{\sqrt{2}}\,\mathrm{e}^{q^2/2b^2-\sqrt{2}wq/b+w^2/2}F(w/\sqrt{2} - q/b), \tag{71}$$

where

$$F(u) = \frac{u}{\sqrt{u^2}}\left[1 - \mathrm{Erf}\left(\frac{u^2}{\sqrt{u^2}}\right)\right] \tag{72}$$

and Erf is the error function. We have used the notation $\sqrt{u^2} = r\exp\left[\mathrm{i}\arctan\left(2\theta\right)/2\right]$ for $u = r\exp\left(\mathrm{i}\theta\right)$, which is simply $|u|$ if u is real. It can be shown, using an integral representation for the Hermite polynomials, that the sum in equation (70) above can also be cast in this form for $\mathrm{Re}(w^2) > 0$. Similar expressions for $f_p(w)$ can be obtained from $f_q(w)$ by replacing q by p and b by $c = \sqrt{m\hbar\omega}$.

5.3. Calculation of matrix elements

Consider the matrix elements

$$\langle z|\hat{X}|z'\rangle = \langle z|\left[\frac{b}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger)\right]|z'\rangle \tag{73}$$

and

$$\langle z|\hat{P}|z'\rangle = \langle z|\left[\frac{c}{i\sqrt{2}}(\hat{a} - \hat{a}^\dagger)\right]|z'\rangle. \tag{74}$$

The transformed functions become

$$f_{\hat{X}|z'\rangle}(w) = \frac{b}{\sqrt{2}}\left(w - \frac{\partial}{\partial w}\right)f_{z'}(w) \tag{75}$$

and

$$f_{\hat{P}|z'\rangle}(w) = \frac{c}{i\sqrt{2}}\left(w + \frac{\partial}{\partial w}\right)f_{z'}(w) \tag{76}$$

where $f_{z'}(w) = 1/(w - z')$.

5.4. Semiclassical limit

In the semiclassical limit, the coherent state propagator $k(z_f^*, z_i, T)$ (which is given by $\langle z_f | \mathrm{e}^{-\mathrm{i}\hat{H}T/\hbar} | z_i \rangle$ for time-independent Hamiltonians) can be written in terms of complex classical trajectories satisfying Hamilton's equations and certain special boundary conditions. Because the trajectories involved are complex, $z(t)$ and $z^*(t)$ are independent classical variables and it is convenient to rename them as $u(t)$ and $v(t)$ respectively. The boundary conditions satisfied by the trajectories contributing to the semiclassical propagator are then given by $u(0) = z_i$, $v(T) = z_f^*$. Using the Weyl symbol $\mathcal{H}$ of the Hamiltonian operator $\hat{H}$ to govern the classical dynamics, the semiclassical approximation for k reads [26–28]

$$k_{sc}(z_f^*, z_i, T) = \sum_{\text{traj.}} \sqrt{\frac{1}{M_{vv}}} \exp\left\{\frac{\mathrm{i}}{\hbar} S\right\}, \tag{77}$$

where S is the action and M_{vv} is an element of the tangent matrix, that propagates small displacements from the trajectory, defined by

$$\begin{pmatrix} \delta u(T) \\ \delta v(T) \end{pmatrix} = \begin{pmatrix} M_{uu} & M_{uv} \\ M_{vu} & M_{vv} \end{pmatrix} \begin{pmatrix} \delta u(0) \\ \delta v(0) \end{pmatrix}. \tag{78}$$

The action satisfies the relations

$$\frac{\partial S}{\partial z_f^*} = -\mathrm{i}\hbar u(T), \qquad \frac{\partial S}{\partial z_i} = -\mathrm{i}\hbar v(0), \qquad \frac{\partial S}{\partial t} = -\mathcal{H}(u(T), z_f^*, t). \tag{79}$$

The conjugate representation of k_{sc} is given, for each contributing trajectory, by

$$\begin{aligned} \tilde{k}_{sc}(w, z_i, T) &= \int_{\tilde{C}} k_{sc}(z_f^*, z_i, T)\, \mathrm{e}^{-z_f^* w}\, \mathrm{d}z_f^* \\ &= \int_{\tilde{C}} \sqrt{\frac{1}{M_{vv}}} \exp\left\{\frac{\mathrm{i}}{\hbar}(S + \mathrm{i}\hbar z_f^* w)\right\} \mathrm{d}z_f^*. \end{aligned} \tag{80}$$

When the integral is performed by the saddle point approximation, the saddle point condition is given by

$$\frac{\partial S}{\partial z_f^*} = -\mathrm{i}\hbar w \tag{81}$$

and the exponent of the transformed expression becomes

$$\tilde{S}(w, z_i, T) = S(z_f, z_i, T) + \mathrm{i}\hbar w z_f^*, \tag{82}$$

where z_f^* is obtained as a function of z_i, w and T from (81). Equations (81) and (82) define a Laplace transformation and comparison with (79) reveals that the trajectory contributing to $\tilde{k}_{sc}$ satisfies $u(0) = z_i$ and $u(T) = w$. When the exponent is expanded to second order around the saddle point and the resulting quadratic integral is performed, the conjugate propagator becomes [26]

$$\tilde{k}_{sc}(w, z_i, T) = \sum_{\text{traj.}} \sqrt{\frac{1}{M_{uv}}} \exp\left\{\frac{\mathrm{i}}{\hbar} \tilde{S}(w, z_i, T)\right\}. \tag{83}$$

The whole conjugation process becomes totally analogous to the conjugation between position and momentum representations. We refer to [26] for the details and for applications related to focal points and the Maslov method.

6. Summary and discussion

The conjugate representation introduced in [24] and studied here in more detail is not standard. The reason for this unconventional approach is that, in contrast to the annihilation operator $\hat{a}$, the creation operator $\hat{a}^\dagger$ does not have eigenstates. However, we have shown that it is still possible to map Bargmann's entire functions $\psi(z^*) = \langle z|\psi\rangle$ into a conjugate set of singular functions $f_\psi(w)$ where the roles of $\hat{a}$ and $\hat{a}^\dagger$ are reversed. The map takes the basis functions $\phi_n = z^{*n}/\sqrt{n!}$ into $f_n = \sqrt{n!}/w^{n+1}$ and a general entire function $\psi(z^*) = \sum_n a_n z^{*n}/\sqrt{n!}$ into $f_\psi(w) = \sum_n a_n\sqrt{n!}/w^{n+1}$.

The conjugate mapping is originally defined by means of a contour integration over a curve γ on the z^* complex plane. The curve is chosen so that $\phi_n(z^*)$ is mapped into $f_n(w)$. However, when applied to a coherent state $|z_0\rangle$, the corresponding integral converges to $1/(w-z_0)$ only if $|w/z_0| > 1$ and the conjugate $f_{z_0}(w)$ has to be analytically continued to the interior of this circle. This continuation has no consequences for the inversion formula, since the integration curve γ' can be chosen to lie outside this region.

We have shown that other formal transformation formulae can be derived which avoid the need of contour integrations, replacing them by integrals over the whole complex plane. These alternative representations, however, are very sensitive to the limited convergence of the line integral defining $f_z(w)$, since they make direct use of this formula. The direct transformation turns out to be only formal, but the inverse transformation formula (49) is well defined and operational.

Acknowledgments

It is a pleasure to thank Hajo Leschke and Alfredo MO de Almeida for interesting discussions. MAMA and ADR acknowledge financial support from CNPq, FAPESP and FINEP. ADR especially acknowledges FAPESP for the fellowship # 04/04614-4 and CNPq for the grant # 474096/2008-4. FP thanks financial support from FACEPE (DCR 0029-1.05/06 and APQ 0800-1.05/06).

Appendix A. The phase-space inversion formula

In this appendix we show how the phase-space inversion formula equation (55) works for the simple cases where $|\psi\rangle = |n\rangle$ and $|\psi\rangle = |z_0\rangle$. The equation is

$$\psi(z^*) = \int w\, \mathrm{e}^{z^* w} f_\psi(w)\, \mathrm{d}^2\mu(w). \tag{A.1}$$

For $|\psi\rangle = |n\rangle$, $f_\psi(w) = \sqrt{n!}/w^{n+1}$ and

$$\psi(z^*) = \sqrt{n!}\int \frac{1}{w^n}\,\mathrm{e}^{z^* w}\,\mathrm{d}^2\mu(w). \tag{A.2}$$

Differentiating with respect to z^* we get

$$\frac{\mathrm{d}^n\psi}{\mathrm{d}z^{*n}} = \sqrt{n!}\int \mathrm{e}^{z^* w}\,\mathrm{d}^2\mu(w) = \sqrt{n!} \tag{A.3}$$

and, therefore, $\psi(z^*) = z^{*n}/\sqrt{n!}$, which is the correct result.

For $|\psi\rangle = |z_0\rangle$ we have $f_\psi(w) = 1/(w - z_0)$ and

$$\psi(z^*) = \int \frac{w}{w - z_0}\, \mathrm{e}^{z^* w}\, \mathrm{d}^2\mu(w) = \int \left(1 + \frac{z_0}{w - z_0}\right) \mathrm{e}^{z^* w}\, \mathrm{d}^2\mu(w)$$
$$= 1 + z_0 \mathrm{e}^{z^* z_0} \int \frac{\mathrm{e}^{z^*(w - z_0)}}{w - z_0}\, \mathrm{d}^2\mu(w) \equiv 1 + z_0\, \mathrm{e}^{z^* z_0} J. \tag{A.4}$$

Since J is an analytic function of z^*,

$$\frac{\mathrm{d}J}{\mathrm{d}z^*} = \int \mathrm{e}^{z^*(w - z_0)}\, \mathrm{d}^2\mu(w) = \mathrm{e}^{-z^* z_0}. \tag{A.5}$$

To integrate this equation back we must be careful with the integration constant. For $z_0 = 0, \mathrm{d}J/\mathrm{d}z^* = 1$ and $J = z^*$, which is the correct result for the ground state $|0\rangle$. The direct integration of (A.5), on the other hand, gives $J = -\mathrm{e}^{-z^* z_0}/z_0$, which does not satisfy the proper condition at $z_0 = 0$. In order to get the correct integration constant we write

$$\frac{\mathrm{d}J}{\mathrm{d}z^*} = \sum_{n=0}^{\infty} \frac{(-1)^n z_0^n z^{*n}}{n!} \tag{A.6}$$

and

$$J = \sum_{n=0}^{\infty} \frac{(-1)^n z_0^n z^{*n+1}}{(n+1)!} = -\frac{1}{z_0} \sum_{n=1}^{\infty} \frac{\left(-z_0^n z^*\right)^n}{n!} = -\frac{1}{z_0}(\mathrm{e}^{-z^* z_0} - 1). \tag{A.7}$$

Substituting back into (A.4) we obtain the correct result $\psi(z^*) = \mathrm{e}^{z^* z_0}$.

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